

Supplementary Appendix

Section S1 provides additional details on the efficient estimation. Section S2 provides the proofs of the results and Section S3 presents additional results on the simulation and empirical analysis.

APPENDIX S1: ADDITIONAL DETAILS ON THE EFFICIENT ESTIMATION

S1.1. Additional intuition about the EIFs in Theorem 2. We focus on $\phi_N(\bar{z}_t) - m_{Y_t S_t}$, which consists of three terms. First, the difference term $m_{Y_t S_t}(z_1, \mathbf{X}_{i1}) - m_{Y_t S_t}$ corresponds to the perturbation of the marginal distribution of $\mathbf{X}_{i1}$. The second term corresponds to the perturbation of the conditional distribution of Y_{it} given $S_{it} = 1$, $\bar{Z}_{it}$, and $\bar{\mathbf{X}}_{it}$. Finally, the last term can be decomposed as:

$$\begin{aligned} & \sum_{t'=2}^t \frac{\mathbf{1}\{\bar{Z}_{i,t'-1} = \bar{z}_{t'-1}\} \{S_{it'} m_{Y_t S_t}(\bar{z}_{t'}, \bar{\mathbf{X}}_{it'}) - S_{i,t'-1} m_{Y_t S_t}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})\}}{\pi_{t'-1}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})} \\ &= \sum_{t'=2}^t \left[\frac{\mathbf{1}\{\bar{Z}_{i,t'-1} = \bar{z}_{t'-1}\} S_{it'}}{\pi_{t'-1}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})} \left\{ m_{Y_t S_t}(\bar{z}_{t'}, \bar{\mathbf{X}}_{it'}) - \frac{m_{Y_t S_t}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})}{p_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})} \right\} \right] \\ & \quad + \sum_{t'=2}^t \left[\frac{\mathbf{1}\{\bar{Z}_{i,t'-1} = \bar{z}_{t'-1}\} \{S_{it'} - S_{i,t'-1} p_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})\} m_{Y_t S_t}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})}{p_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1}) \pi_{t'-1}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})} \right], \end{aligned}$$

where the equality follows in part from $m_{Y_t S_t}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1}) = \mathbb{E}\{m_{Y_t S_t}(\bar{z}_{t'}, \bar{\mathbf{X}}_{it'}) \mid S_{it'} = 1, \bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1}\} \times p_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})$. The first summation corresponds to the perturbation of the conditional distribution of $\bar{\mathbf{X}}_{it'}$ given $S_{it'} = 1$, $\bar{Z}_{i,t'-1}$, and $\bar{\mathbf{X}}_{i,t'-1}$, while the second represents that of the conditional distribution of $S_{it'}$ given $S_{i,t'-1} = 1$, $\bar{Z}_{i,t'-1}$, $\bar{\mathbf{X}}_{i,t'-1}$.

S1.2. Formulas for the efficient estimators. The estimators for $\tau_t(\bar{z}_{t-1})$ and $\theta(\xi)$ are given by

$$\hat{\tau}_t(\bar{z}_{t-1}) = \frac{\hat{\phi}_N(\bar{z}_{t-1}, 1) - \hat{\phi}_N(\bar{z}_{t-1}, 0)}{\hat{\phi}_D(\bar{z}_{t-1})},$$

$$\hat{\theta}(\xi) = \sum_{t=1}^T \sum_{\bar{z}_t} \hat{\phi}_N(\bar{z}_t) \xi(\bar{z}_t),$$

where

$$\begin{aligned} \hat{\phi}_N(\bar{z}_t) &= \frac{1}{n} \sum_{i=1}^n \left[\hat{m}_{Y_t S_t}(z_1, \mathbf{X}_{i1}) + \frac{\mathbf{1}\{Z_{it} = \bar{z}_t\} S_{it} \{Y_{it} - \hat{m}_{Y_t S_t}(\bar{z}_t, \bar{\mathbf{X}}_{it})\}}{\hat{\pi}_t(\bar{z}_t, \bar{\mathbf{X}}_{it})} \right. \\ & \quad \left. + \sum_{t'=2}^t \frac{\mathbf{1}\{\bar{Z}_{i,t'-1} = \bar{z}_{t'-1}\} \{S_{it'} \hat{m}_{Y_t S_t}(\bar{z}_{t'}, \bar{\mathbf{X}}_{it'}) - S_{i,t'-1} \hat{m}_{Y_t S_t}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})\}}{\hat{\pi}_{t'-1}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})} \right], \\ \hat{\phi}_D(\bar{z}_{t-1}) &= \frac{1}{n} \sum_{i=1}^n \left[\hat{m}_{S_t}(z_1, \mathbf{X}_{i1}) + \frac{\mathbf{1}\{\bar{Z}_{i,t-1} = \bar{z}_{t-1}\} \{S_{it} - S_{i,t-1} \hat{m}_{S_t}(\bar{z}_{t-1}, \bar{\mathbf{X}}_{i,t-1})\}}{\hat{\pi}_{t-1}(\bar{z}_{t-1}, \bar{\mathbf{X}}_{i,t-1})} \right. \\ & \quad \left. + \sum_{t'=2}^{t-1} \frac{\mathbf{1}\{\bar{Z}_{i,t'-1} = \bar{z}_{t'-1}\} \{S_{it'} \hat{m}_{S_t}(\bar{z}_{t'}, \bar{\mathbf{X}}_{it'}) - S_{i,t'-1} \hat{m}_{S_t}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})\}}{\hat{\pi}_{t'-1}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})} \right]. \end{aligned}$$

APPENDIX S2: PROOFS

S2.1. Proof of Theorem 1. We first prove Theorem 1(a). We can write

$$\mathbb{E}\{Y_{it}(\bar{z}_t) \mid S_{it}(\bar{z}_{t-1}) = 1\} = \frac{\mathbb{E}\{Y_{it}(\bar{z}_t)S_{it}(\bar{z}_{t-1})\}}{\Pr\{S_{it}(\bar{z}_{t-1}) = 1\}}.$$

We consider the numerator and the denominator separately. For the denominator, because $S_{it} = 1$ implies $S_{it'} = 1$ for any $t \geq t'$, we have,

$$\begin{aligned} & \Pr\{S_{it}(\bar{z}_{t-1}) = 1\} \\ &= \Pr\{S_{it}(\bar{z}_{t-1}) = 1, \dots, S_{i2}(z_1) = 1\} \\ &= \int \Pr\{S_{it}(\bar{z}_{t-1}) = 1, \dots, S_{i2}(z_1) = 1 \mid \mathbf{X}_{i1} = \mathbf{x}_1\} l_1(\mathbf{x}_1) d\mathbf{x}_1 \\ &= \int \Pr\{S_{it}(\bar{z}_{t-1}) = 1, \dots, S_{i2}(z_1) = 1 \mid Z_{i1} = z_1, \mathbf{X}_{i1} = \mathbf{x}_1\} l_1(\mathbf{x}_1) d\mathbf{x}_1 \\ &= \int \Pr\{S_{it}(\bar{z}_{t-1}) = 1, \dots, S_{i3}(\bar{z}_2) = 1 \mid S_{i2} = 1, Z_{i1} = z_1, \mathbf{X}_{i1} = \mathbf{x}_1\} p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\mathbf{x}_1, \end{aligned}$$

where the third equality follows from Assumption 2. We can further write:

$$\begin{aligned} & \Pr\{S_{it}(\bar{z}_{t-1}) = 1, \dots, S_{i3}(\bar{z}_2) = 1 \mid S_{i2} = 1, Z_{i1} = z_1, \mathbf{X}_{i1} = \mathbf{x}_1\} \\ &= \int \Pr\{S_{it}(\bar{z}_{t-1}) = 1, \dots, S_{i3}(\bar{z}_2) = 1 \mid S_{i2} = 1, Z_{i1} = z_1, \bar{\mathbf{X}}_{i2} = \bar{\mathbf{x}}_2\} l_2(z_1, \bar{\mathbf{x}}_2) d\bar{\mathbf{x}}_2 \\ &= \int \Pr\{S_{it}(\bar{z}_{t-1}) = 1, \dots, S_{i3} = 1 \mid S_{i2} = 1, \bar{Z}_{i2} = \bar{z}_2, \bar{\mathbf{X}}_{i2} = \bar{\mathbf{x}}_2\} l_2(z_1, \bar{\mathbf{x}}_2) d\bar{\mathbf{x}}_2 \\ &= \int \Pr\{S_{it}(\bar{z}_{t-1}) = 1, \dots, S_{i4}(\bar{z}_3) = 1 \mid S_{i3} = 1, \bar{Z}_{i2} = \bar{z}_2, \bar{\mathbf{X}}_{i2} = \bar{\mathbf{x}}_2\} p_3(\bar{z}_2, \bar{\mathbf{x}}_2) l_2(z_1, \bar{\mathbf{x}}_2) d\bar{\mathbf{x}}_2, \end{aligned}$$

where the third equality again follows from Assumption 2. Applying this step iteratively, we obtain:

$$(S1) \quad \Pr\{S_{it}(\bar{z}_{t-1}) = 1\} = \int \prod_{t'=2}^t p_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{x}}_{t'-1}) \prod_{t'=1}^{t-1} l_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{x}}_{t'}) d\bar{\mathbf{x}}_{t-1}.$$

For the numerator, we use a similar procedure to obtain:

$$\begin{aligned} & \mathbb{E}\{Y_{it}(\bar{z}_t)S_{it}(\bar{z}_{t-1})\} \\ &= \int \mathbb{E}\{Y_{it}(\bar{z}_t) \mid S_{it} = 1, \bar{Z}_{i,t-1} = \bar{z}_{t-1}, \bar{\mathbf{X}}_{i,t-1} = \bar{\mathbf{x}}_{t-1}\} \prod_{t'=2}^t p_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{x}}_{t'-1}) \prod_{t'=1}^{t-1} l_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{x}}_{t'}) d\bar{\mathbf{x}}_{t-1} \\ &= \int \mathbb{E}\{Y_{it}(\bar{z}_t) \mid S_{it} = 1, \bar{Z}_{i,t-1} = \bar{z}_{t-1}, \bar{\mathbf{X}}_{it} = \bar{\mathbf{x}}_t\} \prod_{t'=2}^t p_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{x}}_{t'-1}) \prod_{t'=1}^t l_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{x}}_{t'}) d\bar{\mathbf{x}}_t \\ &= \int \mu_t(\bar{\mathbf{z}}_t, \bar{\mathbf{x}}_t) \prod_{t'=2}^t p_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{x}}_{t'-1}) \prod_{t'=1}^t l_{t'}(\bar{z}_{t'-1}, \bar{\mathbf{x}}_{t'}) d\bar{\mathbf{x}}_t, \end{aligned}$$

where the third equality follows from Assumption 2. Combining Equations (S1) and (S2) leads to the desired result.

We next prove Theorem 1(b). Using the law of total expectation, we have:

$$\mathbb{E} \left[\frac{\mathbf{1}\{\bar{Z}_{i,t-1} = \bar{z}_{t-1}, S_{it} = 1\}}{\prod_{t'=1}^{t-1} \Pr(Z_{it'} = z_{t'} \mid S_{it'} = 1, \bar{Z}_{i,t'-1} = \bar{z}_{t'-1}, \bar{\mathbf{X}}_{it'} = \bar{\mathbf{x}}_{t'})} \right]$$

$$\begin{aligned}
&= \int \mathbb{E} \left[\frac{\mathbf{1}\{\bar{\mathbf{Z}}_{i,t-1} = \bar{\mathbf{z}}_{t-1}, S_{it} = 1\}}{\prod_{t'=1}^{t-1} \Pr(Z_{it'} = z_{t'} \mid S_{it'} = 1, \bar{\mathbf{Z}}_{i,t'-1} = \bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{X}}_{it'} = \bar{\mathbf{x}}_{t'})} \mid \mathbf{X}_{i1} = \mathbf{x}_1 \right] l_1(\mathbf{x}_1) d\mathbf{x}_1 \\
&= \int \mathbb{E} \left[\frac{\mathbf{1}\{\bar{\mathbf{Z}}_{i,t-1} = \bar{\mathbf{z}}_{t-1}, S_{it} = 1\}}{\prod_{t'=2}^{t-1} \Pr(Z_{it'} = z_{t'} \mid S_{it'} = 1, \bar{\mathbf{Z}}_{i,t'-1} = \bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{X}}_{it'} = \bar{\mathbf{x}}_{t'})} \mid S_{i2} = 1, Z_{i1} = z_1, \mathbf{X}_{i1} = \mathbf{x}_1 \right] \\
&\quad \times p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\mathbf{x}_1 \\
&= \int \mathbb{E} \left[\frac{\mathbf{1}\{\bar{\mathbf{Z}}_{i,t-1} = \bar{\mathbf{z}}_{t-1}, S_{it} = 1\}}{\prod_{t'=2}^{t-1} \Pr(Z_{it'} = z_{t'} \mid S_{it'} = 1, \bar{\mathbf{Z}}_{i,t'-1} = \bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{X}}_{it'} = \bar{\mathbf{x}}_{t'})} \mid S_{i2} = 1, Z_{i1} = z_1, \bar{\mathbf{X}}_{i2} = \bar{\mathbf{x}}_2 \right] \\
&\quad \times p_2(z_1, \mathbf{x}_1) l_2(z_1, \bar{\mathbf{x}}_2) l_1(\mathbf{x}_1) d\bar{\mathbf{x}}_2 \\
&= \int \mathbb{E} \left[\frac{\mathbf{1}\{\bar{\mathbf{Z}}_{i,t-1} = \bar{\mathbf{z}}_{t-1}, S_{it} = 1\}}{\prod_{t'=3}^{t-1} \Pr(Z_{it'} = z_{t'} \mid S_{it'} = 1, \bar{\mathbf{Z}}_{i,t'-1} = \bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{X}}_{it'} = \bar{\mathbf{x}}_{t'})} \mid S_{i3} = 1, \bar{\mathbf{Z}}_{i2} = \bar{\mathbf{z}}_2, \bar{\mathbf{X}}_{i2} = \bar{\mathbf{x}}_2 \right] \\
&\quad \times p_3(\bar{\mathbf{z}}_2, \bar{\mathbf{x}}_2) p_2(z_1, \mathbf{x}_1) l_2(z_1, \bar{\mathbf{x}}_2) l_1(\mathbf{x}_1) d\bar{\mathbf{x}}_2.
\end{aligned}$$

Applying the procedure iteratively, we obtain:

$$\begin{aligned}
&\mathbb{E} \left[\frac{\mathbf{1}\{\mathbf{Z}_{i,t-1} = \bar{\mathbf{z}}_{t-1}, S_{it} = 1\}}{\prod_{t'=1}^{t-1} \Pr(Z_{it'} = z_{t'} \mid S_{it'} = 1, \bar{\mathbf{Z}}_{i,t'-1} = \bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{X}}_{it'} = \bar{\mathbf{x}}_{t'})} \right] \\
&= \int \prod_{t'=2}^t p_{t'}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'-1}) \prod_{t'=1}^{t-1} l_{t'}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'}) d\bar{\mathbf{x}}_{t-1},
\end{aligned}$$

which, together with Equation (S1), yields the inverse probability weighting formula for $\Pr\{S_{it}(\bar{\mathbf{z}}_{t-1}) = 1\}$. Similarly, we can derive:

$$\begin{aligned}
&\mathbb{E} \left[\frac{Y_{it} \mathbf{1}\{\bar{\mathbf{z}}_{it} = \bar{\mathbf{z}}_t, S_{it} = 1\}}{\prod_{t'=1}^t \Pr(Z_{it'} = z_{t'} \mid S_{it'} = 1, \bar{\mathbf{Z}}_{i,t'-1} = \bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{X}}_{it'} = \bar{\mathbf{x}}_{t'})} \right] \\
&= \int \mathbb{E} \left[\frac{Y_{it} \mathbf{1}\{\bar{\mathbf{z}}_{it} = \bar{\mathbf{z}}_t, S_{it} = 1\}}{\Pr(Z_{it} = z_t \mid S_{it} = 1, \bar{\mathbf{z}}_{i,t-1} = \bar{\mathbf{z}}_{t-1}, \bar{\mathbf{X}}_{i,t-1} = \bar{\mathbf{x}}_{t-1})} \mid S_{it} = 1, \bar{\mathbf{Z}}_{i,t-1} = \bar{\mathbf{z}}_{t-1}, \bar{\mathbf{X}}_{i,t-1} = \bar{\mathbf{x}}_{t-1} \right] \\
&\quad \times \prod_{t'=2}^t p_{t'}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'-1}) \prod_{t'=1}^{t-1} l_{t'}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'}) d\bar{\mathbf{x}}_{t-1} \\
&= \int \mu_t(\bar{\mathbf{z}}_t, \bar{\mathbf{x}}_t) \prod_{t'=2}^t p_{t'}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'-1}) \prod_{t'=1}^t l_{t'}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'}) d\bar{\mathbf{x}}_t,
\end{aligned}$$

which, combined with Equation (S2), yields the inverse probability weighting formula for $\mathbb{E}\{Y_{it}(\bar{\mathbf{z}}_t) S_{it}(\bar{\mathbf{z}}_{t-1})\}$. $\square$

S2.2. Proof of Proposition 1. For $m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'})$ with $t' < t$, we have:

$$\begin{aligned}
&m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'}) \\
&= \mathbb{E}\{Y_{it}(\bar{\mathbf{z}}_t) S_{it}(\bar{\mathbf{z}}_{t-1}) \mid S_{i,t'+1} = 1, \bar{\mathbf{Z}}_{it'} = \bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'}\} \Pr(S_{i,t'+1} = 1 \mid S_{it'} = 1, \bar{\mathbf{Z}}_{it'} = \bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'}) \\
&= \mathbb{E} \left[\mathbb{E}\{Y_{it}(\bar{\mathbf{z}}_t) S_{it}(\bar{\mathbf{z}}_{t-1}) \mid S_{i,t'+1} = 1, \bar{\mathbf{Z}}_{it'} = \bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{i,t'+1}\} \mid S_{i,t'+1} = 1, \bar{\mathbf{Z}}_{it'} = \bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'} \right] \\
&\quad \times \Pr(S_{i,t'+1} = 1 \mid S_{it'} = 1, \bar{\mathbf{Z}}_{it'} = \bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'}) \\
&= \mathbb{E} \{m_{Y_t S_t}(\bar{\mathbf{z}}_{t'+1}, \bar{\mathbf{X}}_{i,t'+1}) \mid S_{i,t'+1} = 1, \bar{\mathbf{Z}}_{it'} = \bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'}\} p_{t'+1}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'})
\end{aligned}$$

$$= \mathbb{E} \{ S_{i,t'+1} m_{Y_t S_t}(\bar{z}_{t'+1}, \bar{\mathbf{X}}_{i,t'+1}) \mid S_{it'} = 1, \bar{\mathbf{Z}}_{it'} = \bar{z}_{t'}, \bar{\mathbf{X}}_{it'} \},$$

where the third equality follows from Assumption 2.

For $m_{S_t}(\bar{z}_{t'}, \bar{\mathbf{X}}_{it'})$ with $t' < t - 1$, we have:

$$\begin{aligned} & m_{S_t}(\bar{z}_{t'}, \bar{\mathbf{X}}_{it'}) \\ &= \mathbb{E} \{ S_{it}(\bar{z}_{t-1}) \mid S_{i,t'+1} = 1, \bar{\mathbf{Z}}_{it'} = \bar{z}_{t'}, \bar{\mathbf{X}}_{it'} \} \Pr(S_{i,t'+1} = 1 \mid S_{it'} = 1, \bar{\mathbf{Z}}_{it'} = \bar{z}_{t'}, \bar{\mathbf{X}}_{it'}) \\ &= \mathbb{E} [\mathbb{E} \{ S_{it}(\bar{z}_{t-1}) \mid S_{i,t'+1} = 1, \bar{\mathbf{Z}}_{it'} = \bar{z}_{t'}, \bar{\mathbf{X}}_{i,t'+1} \} \mid S_{i,t'+1} = 1, \bar{\mathbf{Z}}_{it'} = \bar{z}_{t'}, \bar{\mathbf{X}}_{it'}] \\ &\quad \times \Pr(S_{i,t'+1} = 1 \mid S_{it'} = 1, \bar{\mathbf{Z}}_{it'} = \bar{z}_{t'}, \bar{\mathbf{X}}_{it'}) \\ &= \mathbb{E} \{ m_{S_t}(\bar{z}_{t'+1}, \bar{\mathbf{X}}_{i,t'+1}) \mid S_{i,t'+1} = 1, \bar{\mathbf{Z}}_{it'} = \bar{z}_{t'}, \bar{\mathbf{X}}_{it'} \} p_{t'+1}(\bar{z}_{t'}, \bar{\mathbf{X}}_{it'}) \\ &= \mathbb{E} \{ S_{i,t'+1} m_{S_t}(\bar{z}_{t'+1}, \bar{\mathbf{X}}_{i,t'+1}) \mid S_{it'} = 1, \bar{\mathbf{Z}}_{it'} = \bar{z}_{t'}, \bar{\mathbf{X}}_{it'} \}, \end{aligned}$$

where the third equality again follows from Assumption 2. $\square$

S2.3. Proof of Theorem 2. We derive the EIFs for the case with $T = 3$. The derivation with general T is similar and thus omitted. We first introduce the following lemma for the EIF of a ratio-type parameter. The proof of this lemma appears in [Jiang, Yang and Ding \(2022\)](#).

LEMMA S1. Consider a ratio-type parameter $R = N/D$, where N and D are defined as functions of the distribution of variables V . If $\dot{N}_{\theta|_{\theta=0}} = \mathbb{E}\{\varphi_N(V)s(V)\}$ and $\dot{D}_{\theta|_{\theta=0}} = \mathbb{E}\{\varphi_D(V)s(V)\}$ where θ is the parameter of a submodel and $s(\cdot)$ is the score function, then $\dot{R}_{\theta|_{\theta=0}} = \mathbb{E}\{\varphi_R(V)s(V)\}$ where

$$(S3) \quad \varphi_R(V) = \frac{1}{D} \varphi_N(V) - \frac{R}{D} \varphi_D(V).$$

In particular, if $\varphi_N(V)$ and $\varphi_D(V)$ are the EIFs for N and D , then $\varphi_R(V)$ is the EIF for R .

This lemma implies that because

$$\begin{aligned} \tau_3(\bar{z}_2) &= \frac{\mathbb{E}\{Y_{i3}(\bar{z}_2, 1)S_{i3}(\bar{z}_2)\} - \mathbb{E}\{Y_{i3}(\bar{z}_2, 0)S_{i3}(\bar{z}_2)\}}{\mathbb{E}\{S_{i3}(\bar{z}_2)\}}, \\ \theta(\xi) &= \sum_{t=1}^3 \sum_{\bar{z}_t} \mathbb{E}\{Y_{it}(\bar{z}_t)S_{it}(\bar{z}_{t-1})\} \Pr\{S_{it}(\bar{z}_{t-1}) = 1\} \xi_t(\bar{z}_t). \end{aligned}$$

it suffices to derive the EIFs for $\mathbb{E}\{Y_{i3}(\bar{z}_3)S_{i3}(\bar{z}_2)\}$ and $\mathbb{E}\{S_{i3}(\bar{z}_2)\}$. We will use the semi-parametric theory of [Bickel et al. \(1993\)](#) to derive the EIFs. Denote

$$\mathbf{V}_i = (\mathbf{X}_{i1}, Z_{i1}, S_{i2}, S_{i2}\mathbf{X}_{i2}, S_{i2}Z_{i2}, S_{i3}, S_{i3}\mathbf{X}_{i3}, S_{i3}Z_{i3}, S_{i3}Y_{i3}),$$

which consists of all the observed variables up to time $t = 3$. The likelihood of $\mathbf{V}_i$ factorizes as

$$\begin{aligned} f(\mathbf{V}_i) &= f(\mathbf{X}_{i1})f(Z_{i1} \mid \mathbf{X}_{i1})f(S_{i2} \mid Z_{i1}, \mathbf{X}_{i1}) \\ &\quad \times \{f(\mathbf{X}_{i2} \mid S_{i2} = 1, Z_{i1}, \mathbf{X}_{i1})f(Z_{i2} \mid S_{i2} = 1, Z_{i1}, \bar{\mathbf{X}}_{i2})f(S_{i3} \mid S_{i2} = 1, \bar{\mathbf{Z}}_{i2}, \bar{\mathbf{X}}_{i2})\}^{S_{i2}} \\ (S4) \quad &\times \{f(\mathbf{X}_{i3} \mid S_{i3} = 1, \bar{\mathbf{Z}}_{i2}, \bar{\mathbf{X}}_{i2})f(Z_{i3} \mid S_{i2} = 1, \bar{\mathbf{Z}}_{i2}, \bar{\mathbf{X}}_{i3})f(Y_{i3} \mid S_{i3} = 1, \bar{\mathbf{Z}}_{i3}, \bar{\mathbf{X}}_{i3})\}^{S_{i3}}. \end{aligned}$$

To derive the EIF, we consider a one-dimensional parametric submodel $f_{\theta}(\mathbf{V}_i)$, which contains the true model at $\theta = 0$, i.e., $f_{\theta}(\mathbf{V}_i)|_{\theta=0} = f(\mathbf{V}_i)$. We use θ in the subscript to denote the quantities with respect to the submodel, e.g., $\mathbb{E}_{\theta}\{Y_{i3}(\bar{z}_3)S_{i3}(\bar{z}_2)\}$ is the value of $\mathbb{E}\{Y_{i3}(\bar{z}_3)S_{i3}(\bar{z}_2)\}$ in the submodel, and $m_{\theta, Y_3 S_3}(z_1, \mathbf{x}_1)$ is the value of $m_{Y_3 S_3}(z_1, \mathbf{x}_1)$

in the submodel. We use dot to denote the partial derivative with respect to θ , e.g., $\dot{\mathbb{E}}_\theta\{Y_{i3}(\bar{z}_3)S_{i3}(\bar{z}_2)\} = \partial\mathbb{E}_\theta\{Y_{i3}(\bar{z}_3)S_{i3}(\bar{z}_2)\}/\partial\theta$, and use $s_\theta(\cdot)$ to denote the score function of the submodel. From Equation (S4), the score function under the submodel decomposes as

$$\begin{aligned} s_\theta(\mathbf{V}_i) &= \sum_{t=1}^3 S_{it} \cdot s_\theta(\mathbf{X}_{it} \mid S_{it} = 1, \bar{\mathbf{Z}}_{i,t-1}, \bar{\mathbf{X}}_{i,t-1}) + \sum_{t=1}^3 S_{it} \cdot s_\theta(Z_{it} \mid S_{it} = 1, \bar{\mathbf{Z}}_{i,t-1}, \bar{\mathbf{X}}_{it}) \\ &\quad + \sum_{t=2}^3 S_{i,t-1} \cdot s_\theta(S_{it} \mid S_{i,t-1} = 1, \bar{\mathbf{Z}}_{i,t-1}, \bar{\mathbf{X}}_{i,t-1}) + S_{i3} \cdot s_\theta(Y_{i3} \mid S_{i3} = 1, \bar{\mathbf{Z}}_{i3}, \bar{\mathbf{X}}_{i3}), \end{aligned}$$

where $s(\cdot)$ represents the score function with $s_\theta(\cdot)|_{\theta=0} = s(\cdot)$.

From the semiparametric theory, the tangent space is

$$\Lambda = H_1^X \oplus H_2^X \oplus H_3^X \oplus H_1^Z \oplus H_2^Z \oplus H_3^Z \oplus H_2^S \oplus H_3^S \oplus H_3^Y,$$

which is the direct sum of orthogonal subspaces

$$\begin{aligned} H_t^X &= \{S_{it}h(\bar{\mathbf{Z}}_{i,t-1}, \bar{\mathbf{X}}_{it}) : \mathbb{E}\{h(\bar{\mathbf{Z}}_{i,t-1}, \bar{\mathbf{X}}_{it}) \mid S_{it} = 1, \bar{\mathbf{Z}}_{i,t-1}, \bar{\mathbf{X}}_{i,t-1}\} = 0\}, \\ H_t^Z &= \{S_{it}h(\bar{\mathbf{Z}}_{it}, \bar{\mathbf{X}}_{it}) : \mathbb{E}\{h(\bar{\mathbf{Z}}_{it}, \bar{\mathbf{X}}_{it}) \mid S_{it} = 1, \bar{\mathbf{Z}}_{i,t-1}, \bar{\mathbf{X}}_{it}\} = 0\}, \\ H_2^S &= \{h(S_{i2}, Z_{i1}, \mathbf{X}_{i1}) : \mathbb{E}\{h(S_{i2}, Z_{i1}, \mathbf{X}_{i1}) \mid Z_{i1}, \mathbf{X}_{i1}\} = 0\}, \\ H_3^S &= \{S_{i2}h(S_{i3}, \bar{\mathbf{Z}}_{i2}, \bar{\mathbf{X}}_{i2}) : \mathbb{E}\{h(S_{i3}, \bar{\mathbf{Z}}_{i2}, \bar{\mathbf{X}}_{i2}) \mid S_{i2} = 1, \bar{\mathbf{Z}}_{i2}, \bar{\mathbf{X}}_{i2}\} = 0\}, \\ H_3^Y &= \{S_{i3}h(Y_{i3}, \bar{\mathbf{Z}}_{i3}, \bar{\mathbf{X}}_{i3}) : \mathbb{E}\{h(Y_{i3}, \bar{\mathbf{Z}}_{i3}, \bar{\mathbf{X}}_{i3}) \mid S_{i3} = 1, \bar{\mathbf{Z}}_{i3}, \bar{\mathbf{X}}_{i3}\} = 0\}. \end{aligned}$$

The EIF for $m_{Y_3S_3} = \mathbb{E}\{Y_{i3}(\bar{z}_3)S_{i3}(\bar{z}_2)\}$, denoted by $\phi_1(\mathbf{V}_i)$, must satisfy

$$\dot{m}_{\theta, Y_3S_3}|_{\theta=0} = \mathbb{E}\{\phi_1(\mathbf{V}_i)s(\mathbf{V}_i)\}$$

and $\phi_1(\mathbf{V}_i) \in \Lambda$. From the chain rule, we have

$$\dot{m}_{\theta, Y_3S_3}|_{\theta=0} = B_1 + B_2,$$

where

$$\begin{aligned} B_1 &= \int m_{Y_3S_3}(z_1, \mathbf{x}_1) \dot{l}_{1\theta}(\mathbf{x}_1)|_{\theta=0} d\mathbf{x}_1 = \int m_{Y_3S_3}(z_1, \mathbf{x}_1) s(\mathbf{x}_1) l_1(\mathbf{x}_1) d\mathbf{x}_1 \\ &= \mathbb{E}\{m_{Y_3S_3}(z_1, \mathbf{X}_{i1}) s(\mathbf{X}_{i1})\}, \\ B_2 &= \int \dot{m}_{\theta, Y_3S_3}(z_1, \mathbf{x}_1)|_{\theta=0} l_1(\mathbf{x}_1) d\mathbf{x}_1. \end{aligned}$$

For B_2 , we can further decompose it as $B_2 = C_1 + C_2$, where

$$\begin{aligned} C_1 &= \int \mathbb{E}\{Y_{i3}(\bar{z}_3)S_{i3}(\bar{z}_2) \mid S_{i2} = 1, Z_{i1} = z_1, \mathbf{X}_{i1} = \mathbf{x}_1\} \dot{p}_{2\theta}(z_1, \mathbf{x}_1)|_{\theta=0} l_1(\mathbf{x}_1) d\mathbf{x}_1, \\ C_2 &= \int \dot{\mathbb{E}}_\theta\{Y_{i3}(\bar{z}_3)S_{i3}(\bar{z}_2) \mid S_{i2} = 1, Z_{i1} = z_1, \mathbf{X}_{i1} = \mathbf{x}_1\}|_{\theta=0} p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\mathbf{x}_1. \end{aligned}$$

For C_1 , we have

$$\begin{aligned} C_1 &= \int \mathbb{E}\{Y_{i3}(\bar{z}_3)S_{i3}(\bar{z}_2) \mid S_{i2} = 1, Z_{i1} = z_1, \mathbf{X}_{i1} = \mathbf{x}_1\} s(S_{i2} = 1 \mid z_1, \mathbf{x}_1) p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\mathbf{x}_1 \\ &= \int \mathbb{E}\left[\frac{\mathbf{1}\{Z_{i1} = z_1\} S_{i2} \mathbb{E}\{Y_{i3}(\bar{z}_3)S_{i3}(\bar{z}_2) \mid S_{i2} = 1, Z_{i1}, \mathbf{X}_{i1} = \mathbf{x}_1\}}{\pi_1(z_1, \mathbf{X}_{i1})} \cdot s(S_{i2} \mid Z_{i1}, \mathbf{X}_{i1}) \mid \mathbf{X}_{i1} = \mathbf{x}_1\right] \\ &\quad \times l_1(\mathbf{x}_1) d\mathbf{x}_1 \end{aligned}$$

$$= \mathbb{E} \left[\frac{\mathbf{1}\{Z_{i1} = z_1\} S_{i2} \mathbb{E}\{Y_{i3}(\bar{z}_3) S_{i3}(\bar{z}_2) \mid S_{i2} = 1, Z_{i1}, \mathbf{X}_{i1}\}}{\pi_1(z_1, \mathbf{X}_{i1})} \cdot s(S_{i2} \mid Z_{i1}, \mathbf{X}_{i1}) \right],$$

where we simplify $s(S_{i2} = 1 \mid Z_{i1} = z_1, \mathbf{X}_{i1} = \mathbf{x}_1)$ as $s(S_{i2} = 1 \mid z_1, \mathbf{x}_1)$. For C_2 , we can decompose $C_2 = D_1 + D_2$, where

$$D_1 = \int m_{Y_3 S_3}(\bar{z}_2, \bar{x}_2) \dot{l}_{2\theta}(z_1, \bar{x}_2) |_{\theta=0} p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\bar{x}_2,$$

$$D_2 = \int \dot{m}_{\theta, Y_3 S_3}(\bar{z}_2, \bar{x}_2) |_{\theta=0} l_2(z_1, \bar{x}_2) p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\bar{x}_2.$$

For D_1 , we have

$$\begin{aligned} D_1 &= \int m_{Y_3 S_3}(\bar{z}_2, \bar{x}_2) s(\mathbf{X}_{i2} = \mathbf{x}_2 \mid S_{i2} = 1, z_1, \mathbf{x}_1) l_2(z_1, \bar{x}_2) p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\bar{x}_2, \\ &= \int \mathbb{E} \{ m_{Y_3 S_3}(\bar{z}_2, \mathbf{x}_1, \mathbf{X}_{i2}) s(\mathbf{X}_{i2} \mid S_{i2} = 1, z_1, \mathbf{x}_1) \mid S_{i2} = 1, z_1, \mathbf{x}_1 \} p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\mathbf{x}_1 \\ &= \int \mathbb{E} \left[\frac{\mathbf{1}\{Z_{i1} = z_1\} S_{i2} m_{Y_3 S_3}(\bar{z}_2, \mathbf{x}_1, \mathbf{X}_{i2})}{\pi_1(z_1, \mathbf{X}_{i1})} \cdot s(\mathbf{X}_{i2} \mid S_{i2} = 1, Z_{i1}, \mathbf{x}_1) \mid \mathbf{X}_{i1} = \mathbf{x}_1 \right] l_1(\mathbf{x}_1) d\mathbf{x}_1 \\ &= \mathbb{E} \left[\frac{\mathbf{1}\{Z_{i1} = z_1\} S_{i2} m_{Y_3 S_3}(z_2, \mathbf{X}_{i2})}{\pi_1(z_1, \mathbf{X}_{i1})} \cdot s(\mathbf{X}_{i2} \mid S_{i2} = 1, Z_{i1}, \mathbf{X}_{i1}) \right], \end{aligned}$$

For D_2 , we have $D_2 = E_1 + E_2$, where

$$E_1 = \int \mathbb{E} \{ Y_{i3}(\bar{z}_3) \mid S_{i3} = 1, \bar{Z}_{i2} = \bar{z}_2, \bar{\mathbf{X}}_{i2} = \bar{x}_2 \} \dot{p}_{3\theta}(\bar{z}_2, \bar{x}_2) |_{\theta=0} l_2(z_1, \bar{x}_2) p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\bar{x}_2,$$

$$E_2 = \int \dot{\mathbb{E}}_{\theta} \{ Y_{i3}(\bar{z}_3) \mid S_{i3} = 1, \bar{Z}_{i2} = \bar{z}_2, \bar{\mathbf{X}}_{i2} = \bar{x}_2 \} |_{\theta=0} p_3(\bar{z}_2, \bar{x}_2) l_2(z_1, \bar{x}_2) p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\bar{x}_2.$$

For E_1 , we have

$$\begin{aligned} E_1 &= \int \mathbb{E} \{ Y_{i3}(\bar{z}_3) \mid S_{i3} = 1, \bar{Z}_{i1} = \bar{z}_2, \bar{\mathbf{X}}_{i2} = \bar{x}_2 \} s(S_{i3} = 1 \mid S_{i2} = 1, \bar{z}_2, \bar{x}_2) \\ &\quad \times p_3(\bar{z}_2, \bar{x}_2) l_2(z_1, \bar{x}_2) p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\bar{x}_2 \\ &= \mathbb{E} \left[\frac{\mathbf{1}\{\bar{Z}_{i2} = \bar{z}_2\} S_{i3} \mathbb{E}\{Y_{i3}(\bar{z}_3) S_{i3}(\bar{z}_2) \mid S_{i3} = 1, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}\}}{\pi_2(\bar{z}_2, \bar{\mathbf{X}}_{i2})} \cdot s(S_{i3} = 1 \mid S_{i2} = 1, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}) \right]. \end{aligned}$$

For E_2 , we have $E_2 = F_1 + F_2$, where we use the double integral to denote multiple integrals,

$$\begin{aligned} F_1 &= \int m_{Y_3 S_3}(\bar{z}_3, \bar{x}_3) \dot{l}_{3\theta}(\bar{z}_2, \bar{x}_3) |_{\theta=0} p_3(\bar{z}_2, \bar{x}_2) l_2(z_1, \bar{x}_2) p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\bar{x}_3 \\ &= \int m_{Y_3 S_3}(\bar{z}_3, \bar{x}_3) s(\mathbf{X}_{i3} = \mathbf{x}_3 \mid S_{i3} = 1, \bar{z}_2, \bar{x}_2) l_3(\bar{z}_2, \bar{x}_3) p_3(\bar{z}_2, \bar{x}_2) l_2(z_1, \bar{x}_2) p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\bar{x}_3 \\ &= \mathbb{E} \left[\frac{\mathbf{1}\{\bar{Z}_{i2} = \bar{z}_2\} S_{i3} m_{Y_3 S_3}(\bar{z}_3, \bar{\mathbf{X}}_{i3})}{\pi_2(\bar{z}_2, \bar{\mathbf{X}}_{i2})} \cdot s(\mathbf{X}_{i3} \mid S_{i3} = 1, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}) \right], \\ F_2 &= \int \dot{m}_{\theta, Y_3 S_3}(\bar{z}_3, \bar{x}_3) |_{\theta=0} l_3(\bar{z}_2, \bar{x}_3) p_3(\bar{z}_2, \bar{x}_2) l_2(z_1, \bar{x}_2) p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\bar{x}_3. \end{aligned}$$

For F_2 , we have

$$\begin{aligned} \dot{m}_{\theta, Y_3 S_3}(\bar{z}_3, \bar{x}_3) |_{\theta=0} &= \dot{\mathbb{E}}_{\theta}(Y_{i3} \mid S_{i3} = 1, \bar{Z}_{i3} = \bar{z}_3, \bar{\mathbf{X}}_{i3} = \bar{x}_3) |_{\theta=0} \\ &= \mathbb{E}\{Y_{i3} \cdot s(Y_{i3} \mid S_{i3} = 1, \bar{Z}_{i3} = \bar{z}_3, \bar{\mathbf{X}}_{i3} = \bar{x}_3) \mid S_{i3} = 1, \bar{Z}_{i3} = \bar{z}_3, \bar{\mathbf{X}}_{i3} = \bar{x}_3\}. \end{aligned}$$

Therefore, we obtain

$$F_2 = \mathbb{E} \left[\frac{\mathbf{1}\{\bar{Z}_{i3} = \bar{z}_3\} S_{i3} Y_{i3}}{\pi_3(\bar{z}_3, \bar{\mathbf{X}}_{i3})} \cdot s(Y_{i3} | S_{i3} = 1, \bar{Z}_{i3}, \bar{\mathbf{X}}_{i3}) \right].$$

Combining all the results, we obtain

$$\begin{aligned} & \dot{\mathbb{E}}_\theta \{Y_{i3}(\bar{z}_3) S_{i3}(\bar{z}_2)\} |_{\theta=0} \\ &= \mathbb{E}\{h_1(\mathbf{X}_{i1}) s(\mathbf{X}_{i1})\} + \mathbb{E}\{h_2(S_{i2}, Z_{i1}, \mathbf{X}_{i1}) s(S_{i2} | Z_{i1}, \mathbf{X}_{i1})\} \\ & \quad + \mathbb{E}\{h_3(S_{i2}, Z_{i1}, \bar{\mathbf{X}}_{i2}) s(\mathbf{X}_{i2} | S_{i2} = 1, Z_{i1}, \mathbf{X}_{i1})\} + \mathbb{E}\{h_4(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}) s(S_{i3} | S_{i2} = 1, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2})\} \\ & \quad + \mathbb{E}\{h_5(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i3}) s(\mathbf{X}_{i3} | S_{i3} = 1, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2})\} + \mathbb{E}\{h_6(Y_{i3}, S_{i3}, \bar{Z}_{i3}, \bar{\mathbf{X}}_{i3}) s(Y_{i3} | S_{i3} = 1, \bar{Z}_{i3}, \bar{\mathbf{X}}_{i3})\}, \end{aligned}$$

where

$$\begin{aligned} h_1(\mathbf{X}_{i1}) &= m_{Y_3 S_3}(z_1, \mathbf{X}_{i1}), \\ h_2(S_{i2}, Z_{i1}, \mathbf{X}_{i1}) &= \frac{\mathbf{1}\{Z_{i1} = z_1\} S_{i2} \mathbb{E}\{Y_{i3}(\bar{z}_3) S_{i3}(\bar{z}_2) | S_{i2} = 1, Z_{i1}, \mathbf{X}_{i1}\}}{\pi_1(z_1, \mathbf{X}_{i1})}, \\ h_3(S_{i2}, Z_{i1}, \bar{\mathbf{X}}_{i2}) &= \frac{\mathbf{1}\{Z_{i1} = z_1\} S_{i2} m_{Y_3 S_3}(\bar{z}_2, \bar{\mathbf{X}}_{i2})}{\pi_1(z_1, \mathbf{X}_{i1})}, \\ h_4(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}) &= \frac{\mathbf{1}\{\bar{Z}_{i2} = \bar{z}_2\} S_{i3} \mathbb{E}\{Y_{i3}(\bar{z}_3) S_{i3}(\bar{z}_2) | S_{i3} = 1, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}\}}{\pi_2(\bar{z}_2, \bar{\mathbf{X}}_{i2})}, \\ h_5(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i3}) &= \frac{\mathbf{1}\{\bar{Z}_{i2} = \bar{z}_2\} S_{i3} m_{Y_3 S_3}(\bar{z}_3, \bar{\mathbf{X}}_{i3})}{\pi_2(\bar{z}_2, \bar{\mathbf{X}}_{i2})}, \\ h_6(Y_{i3}, S_{i3}, \bar{Z}_{i3}, \bar{\mathbf{X}}_{i3}) &= \frac{\mathbf{1}\{\bar{Z}_{i3} = \bar{z}_3\} S_{i3} Y_{i3}}{\pi_3(\bar{z}_3, \bar{\mathbf{X}}_{i3})}. \end{aligned}$$

Denote

$$\begin{aligned} \phi_{N1}(\mathbf{X}_{i1}) &= h_1(\mathbf{X}_{i1}) - \mathbb{E}\{h_1(\mathbf{X}_{i1})\}, \\ \phi_{N2}(S_{i2}, Z_{i1}, \mathbf{X}_{i1}) &= h_2(S_{i2}, Z_{i1}, \mathbf{X}_{i1}) - \mathbb{E}\{h_2(S_{i2}, Z_{i1}, \mathbf{X}_{i1}) | Z_{i1}, \mathbf{X}_{i1}\}, \\ \phi_{N3}(S_{i2}, Z_{i1}, \bar{\mathbf{X}}_{i2}) &= h_3(S_{i2}, Z_{i1}, \bar{\mathbf{X}}_{i2}) - \mathbb{E}\{h_3(S_{i2}, Z_{i1}, \bar{\mathbf{X}}_{i2}) | S_{i2} = 1, Z_{i1}, \mathbf{X}_{i1}\}, \\ \phi_{N4}(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}) &= h_4(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}) - \mathbb{E}\{h_4(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}) | S_{i2} = 1, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}\}, \\ \phi_{N5}(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i3}) &= h_5(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i3}) - \mathbb{E}\{h_5(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i3}) | S_{i3} = 1, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}\}, \\ \phi_{N6}(Y_{i3}, S_{i3}, \bar{Z}_{i3}, \bar{\mathbf{X}}_{i3}) &= h_6(Y_{i3}, S_{i3}, \bar{Z}_{i3}, \bar{\mathbf{X}}_{i3}) - \mathbb{E}\{h_6(Y_{i3}, S_{i3}, \bar{Z}_{i3}, \bar{\mathbf{X}}_{i3}) | S_{i3} = 1, \bar{Z}_{i3}, \bar{\mathbf{X}}_{i3}\}. \end{aligned}$$

We can verify that $\phi_{N1}(\mathbf{X}_{i1}) \in H_1^{\mathbf{X}}$, $\phi_{N2}(S_{i2}, Z_{i1}, \mathbf{X}_{i1}) \in H_2^S$, $\phi_{N3}(S_{i2}, Z_{i1}, \bar{\mathbf{X}}_{i2}) \in H_2^{\mathbf{X}}$, $\phi_{N4}(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}) \in H_3^S$, $\phi_{N5}(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i3}) \in H_3^{\mathbf{X}}$, and $\phi_{N6}(Y_{i3}, S_{i3}, \bar{Z}_{i3}, \bar{\mathbf{X}}_{i3}) \in H_3^Y$. Therefore,

$$\dot{\mathbb{E}}_\theta \{Y_{i3}(\bar{z}_3) S_{i3}(\bar{z}_2)\} |_{\theta=0} = \mathbb{E}\{\phi_N(\mathbf{V}_i) s(\mathbf{V}_i)\},$$

where

$$\begin{aligned} & \phi_N(\bar{\mathbf{z}}_t) \\ &= \phi_{N1}(\mathbf{X}_{i1}) + \phi_{N2}(S_{i2}, Z_{i1}, \mathbf{X}_{i1}) + \phi_{N3}(S_{i2}, Z_{i1}, \bar{\mathbf{X}}_{i2}) + \phi_{N4}(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i2}) \\ & \quad + \phi_{N5}(S_{i3}, \bar{Z}_{i2}, \bar{\mathbf{X}}_{i3}) + \phi_{N6}(Y_{i3}, S_{i3}, \bar{Z}_{i3}, \bar{\mathbf{X}}_{i3}) \\ &= \frac{\mathbf{1}\{\bar{Z}_{i3} = \bar{z}_3\} S_{i3} \cdot \{Y_{i3} - m_{Y_3 S_3}(\bar{z}_3, \bar{\mathbf{X}}_{i3})\}}{\pi_3(\bar{z}_3, \bar{\mathbf{X}}_{i3})} + \frac{\mathbf{1}\{\bar{Z}_{i2} = \bar{z}_2\} S_{i3}}{\pi_2(\bar{z}_2, \bar{\mathbf{X}}_{i2})} \cdot \left\{ m_{Y_3 S_3}(\bar{z}_3, \bar{\mathbf{X}}_{i3}) - \frac{m_{Y_3 S_3}(\bar{z}_2, \bar{\mathbf{X}}_{i2})}{p_3(\bar{z}_2, \bar{\mathbf{X}}_{i2})} \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{\mathbf{1}\{\bar{Z}_{i2} = \bar{z}_2\} \{S_{i3} - S_{i2}p_3(\bar{z}_2, \bar{\mathbf{X}}_{i2})\} \cdot m_{Y_3S_3}(\bar{z}_2, \bar{\mathbf{X}}_{i2})}{p_3(\bar{z}_2, \bar{\mathbf{X}}_{i2})\pi_2(\bar{z}_2, \bar{\mathbf{X}}_{i2})} \\
& + \frac{\mathbf{1}\{Z_{i1} = z_1\}S_{i2}}{\pi_1(z_1, \mathbf{X}_{i1})} \cdot \left\{ m_{Y_3S_3}(\bar{z}_2, \bar{\mathbf{X}}_{i2}) - \frac{m_{Y_3S_3}(z_1, \mathbf{X}_{i1})}{p_2(z_1, \mathbf{X}_{i1})} \right\} \\
& + \frac{\mathbf{1}\{Z_{i1} = z_1\} \{S_{i2} - p_2(z_1, \mathbf{X}_{i1})\} \cdot m_{Y_3S_3}(z_1, \mathbf{X}_{i1})}{p_2(z_1, \mathbf{X}_{i1})\pi_1(\mathbf{X}_{i1})} + m_{Y_3S_3}(z_1, \mathbf{X}_{i1}) - m_{Y_3S_3} \\
& = \frac{\mathbf{1}\{\bar{Z}_{i3} = \bar{z}_3\}S_{i3} \cdot \{Y_{i3} - m_{Y_3S_3}(\bar{z}_3, \bar{\mathbf{X}}_{i3})\}}{\pi_3(\bar{z}_3, \bar{\mathbf{X}}_{i3})} + \frac{\mathbf{1}\{\bar{Z}_{i2} = \bar{z}_2\} \{S_{i3}m_{Y_3S_3}(\bar{z}_3, \bar{\mathbf{X}}_{i3}) - S_{i2}m_{Y_3S_3}(\bar{z}_2, \bar{\mathbf{X}}_{i2})\}}{\pi_2(\bar{z}_2, \bar{\mathbf{X}}_{i2})} \\
& + \frac{\mathbf{1}\{Z_{i1} = z_1\} \cdot \{S_{i2}m_{Y_3S_3}(\bar{z}_2, \bar{\mathbf{X}}_{i2}) - m_{Y_3S_3}(z_1, \mathbf{X}_{i1})\}}{\pi_1(z_1, \mathbf{X}_{i1})} + m_{Y_3S_3}(z_1, \mathbf{X}_{i1}) - m_{Y_3S_3}.
\end{aligned}$$

Similarly, we can show that

$$\dot{\mathbb{E}}_\theta\{S_{i3}(\bar{z}_2)\}_{|\theta=0} = \mathbb{E}\{\phi_D(\mathbf{V}_i)s(\mathbf{V}_i)\},$$

where

$$\begin{aligned}
\phi_D(\bar{z}_{t-1}) & = \frac{\mathbf{1}\{\bar{Z}_{i2} = \bar{z}_2\} \{S_{i3} - S_{i2}m_{S_3}(\bar{z}_2, \bar{\mathbf{X}}_{i2})\}}{\pi_2(\bar{z}_2, \bar{\mathbf{X}}_{i2})} + \frac{\mathbf{1}\{Z_{i1} = z_1\}S_{i2}}{\pi_1(z_1, \mathbf{X}_{i1})} \cdot \left\{ m_{S_3}(\bar{z}_2, \bar{\mathbf{X}}_{i2}) - \frac{m_{S_3}(z_1, \mathbf{X}_{i1})}{p_2(z_1, \mathbf{X}_{i1})} \right\} \\
& + \frac{\mathbf{1}\{Z_{i1} = z_1\} \{S_{i2} - p_2(z_1, \mathbf{X}_{i1})\} \cdot m_{S_3}(z_1, \mathbf{X}_{i1})}{p_2(z_1, \mathbf{X}_{i1})\pi_1(z_1, \mathbf{X}_{i1})} + m_{S_3}(z_1, \mathbf{X}_{i1}) - m_{S_3} \\
& = \frac{\mathbf{1}\{\bar{Z}_{i2} = \bar{z}_2\} \{S_{i3} - S_{i2}m_{S_3}(\bar{z}_2, \bar{\mathbf{X}}_{i2})\}}{\pi_2(\bar{z}_2, \bar{\mathbf{X}}_{i2})} + \frac{\mathbf{1}\{Z_{i1} = z_1\} \{S_{i2}m_{S_3}(\bar{z}_2, \bar{\mathbf{X}}_{i2}) - m_{S_3}(z_1, \mathbf{X}_{i1})\}}{\pi_1(z_1, \mathbf{X}_{i1})} \\
& + m_{S_3}(z_1, \mathbf{X}_{i1}) - m_{S_3}.
\end{aligned}$$

Therefore, the EIF for $\tau_3(\bar{z}_2)$ is

$$\frac{\phi_N(\bar{z}_2, 1) - \phi_N(\bar{z}_2, 0) - \tau_t(\bar{z}_2) \cdot \phi_D(\bar{z}_2)}{\Pr\{S_{it}(\bar{z}_2) = 1\}}$$

and the EIF for $\theta(\xi)$ is

$$\sum_{t=1}^T \sum_{\bar{\mathbf{z}}_t} \phi_N(\bar{\mathbf{z}}_t) \xi_t(\bar{\mathbf{z}}_t) - \theta(\xi).$$

□

S2.4. Proof of Theorem 3. The estimator $\hat{\phi}_N(\bar{\mathbf{z}}_t)$ is consistent for

$$\begin{aligned}
& \mathbb{E} \left[\frac{\mathbf{1}\{\bar{Z}_{it} = \bar{z}_t\} S_{it} \{Y_{it} - \tilde{m}_{Y_t S_t}(\bar{z}_{it}, \bar{\mathbf{X}}_{it})\}}{\tilde{\pi}_t(\bar{z}_t, \bar{\mathbf{X}}_{it})} \right] \\
& + \sum_{t'=2}^t \mathbb{E} \left[\frac{\mathbf{1}\{\bar{Z}_{i,t'-1} = \bar{z}_{t'-1}\} \{S_{it'} \tilde{m}_{Y_t S_t}(\bar{z}_{t'}, \bar{\mathbf{X}}_{it'}) - S_{i,t'-1} \tilde{m}_{Y_t S_t}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})\}}{\tilde{\pi}_{t'-1}(\bar{z}_{t'-1}, \bar{\mathbf{X}}_{i,t'-1})} \right] \\
& \text{(S5)}
\end{aligned}$$

We compute the terms in Equation (S5) separately. For any t' , we have

$$\mathbb{E} \left[\frac{\mathbf{1}\{\bar{Z}_{it'} = \bar{z}_{t'}\} S_{it'} \tilde{m}_{Y_t S_t}(\bar{z}_{t'}, \bar{\mathbf{X}}_{it'})}{\tilde{\pi}_{t'}(\bar{z}_{t'}, \bar{\mathbf{X}}_{it'})} \right]$$

$$\begin{aligned}
&= \int \mathbb{E} \left[\frac{\mathbf{1}\{\bar{\mathbf{Z}}_{it'} = \bar{\mathbf{z}}_{t'}\} S_{it'} \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'})}{\tilde{\pi}_{t'-1}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'})} \mid \mathbf{X}_{i1} = \mathbf{x}_1 \right] l_1(\mathbf{x}_1) d\mathbf{x}_1 \\
&= \int \mathbb{E} \left[\frac{\mathbf{1}\{\bar{\mathbf{Z}}_{it'} = \bar{\mathbf{z}}_{t'}\} S_{it'} \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'}) w_1(z_1, \mathbf{x}_1)}{\tilde{\pi}_{t'}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'})} \mid Z_{i1} = z_1, \mathbf{X}_{i1} = \mathbf{x}_1 \right] l_1(\mathbf{x}_1) d\mathbf{x}_1 \\
&= \int \mathbb{E} \left[\frac{\mathbf{1}\{\bar{\mathbf{Z}}_{it'} = \bar{\mathbf{z}}_{t'}\} S_{it'} \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'}) w_1(z_1, \mathbf{x}_1)}{\tilde{\pi}_{t'}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'})} \mid S_{i2} = 1, Z_{i1} = z_1, \mathbf{X}_{i1} = \mathbf{x}_1 \right] p_2(z_1, \mathbf{x}_1) l_1(\mathbf{x}_1) d\mathbf{x}_1,
\end{aligned}$$

where the first equality follows from the law of total expectation. By applying this calculation iteratively, we obtain

$$\begin{aligned}
&\mathbb{E} \left[\frac{\mathbf{1}\{\bar{\mathbf{Z}}_{it'} = \bar{\mathbf{z}}_{t'}\} S_{it'} \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'})}{\tilde{\pi}_{t'}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'})} \right] \\
&= \int \frac{\tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \pi_{t'}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'})}{\tilde{\pi}_{t'}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'})} \prod_{s=2}^{t'} p_s(\bar{\mathbf{z}}_{s-1}, \bar{\mathbf{x}}_{s-1}) \prod_{s=1}^{t'} l_s(\bar{\mathbf{z}}_{s-1}, \bar{\mathbf{x}}_s) d\bar{\mathbf{x}}_{t'},
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mathbb{E} \left[\frac{\mathbf{1}\{\bar{\mathbf{Z}}_{i,t'-1} = \bar{\mathbf{z}}_{t'-1}\} S_{i,t'} \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{X}}_{it'})}{\tilde{\pi}_{t'-1}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{X}}_{it'-1})} \right] \\
&= \int \frac{\tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \pi_{t'-1}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'-1})}{\tilde{\pi}_{t'-1}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'-1})} \prod_{s=2}^{t'} p_s(\bar{\mathbf{z}}_{s-1}, \bar{\mathbf{x}}_{s-1}) \prod_{s=1}^{t'} l_s(\bar{\mathbf{z}}_{s-1}, \bar{\mathbf{x}}_s) d\bar{\mathbf{x}}_{t'}, \\
&\mathbb{E} \left[\frac{\mathbf{1}\{\bar{\mathbf{Z}}_{it} = \bar{\mathbf{z}}_t\} S_{it} Y_{it}}{\tilde{\pi}_t(\bar{\mathbf{z}}_t, \bar{\mathbf{X}}_{it})} \right] \\
&= \int \frac{m_{Y_t S_t}(\bar{\mathbf{z}}_t, \bar{\mathbf{x}}_t) \pi_t(\bar{\mathbf{z}}_t, \bar{\mathbf{x}}_t)}{\tilde{\pi}_t(\bar{\mathbf{z}}_t, \bar{\mathbf{x}}_t)} \prod_{s=2}^t p_s(\bar{\mathbf{z}}_{s-1}, \bar{\mathbf{x}}_{s-1}) \prod_{s=1}^t l_s(\bar{\mathbf{z}}_{s-1}, \bar{\mathbf{x}}_s) d\bar{\mathbf{x}}_t.
\end{aligned}$$

For simplicity, we omit the dependency of $p_s(\bar{\mathbf{z}}_{s-1}, \bar{\mathbf{x}}_{s-1})$, $l_s(\bar{\mathbf{z}}_{s-1}, \bar{\mathbf{x}}_s)$, and $\pi_t(\bar{\mathbf{z}}_t, \bar{\mathbf{x}}_t)$ on $\bar{\mathbf{z}}$ and $\bar{\mathbf{x}}$ and write them as p_s , l_s , and π_t , respectively. We can then write Equation (S5) as

$$\begin{aligned}
&\int \frac{\{m_{Y_t S_t}(\bar{\mathbf{z}}_t, \bar{\mathbf{x}}_t) - \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_t, \bar{\mathbf{x}}_t)\} \pi_t}{\tilde{\pi}_t} \prod_{s=2}^t p_s \prod_{s=1}^t l_s d\bar{\mathbf{x}}_t \\
&+ \sum_{t'=2}^t \left[\int \frac{\tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \pi_{t'-1}}{\tilde{\pi}_{t'-1}} \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} - \int \frac{\tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'-1}) \pi_{t'-1}}{\tilde{\pi}_{t'-1}} \prod_{s=2}^{t'-1} p_s \prod_{s=1}^{t'-1} l_s d\bar{\mathbf{x}}_{t'-1} \right] \\
&+ \int \tilde{m}_{Y_t S_t}(z_1, \mathbf{x}_1) l_1 d\mathbf{x}_1 \\
&= \int \frac{m_{Y_t S_t}(\bar{\mathbf{z}}_t, \bar{\mathbf{x}}_t) \pi_t}{\tilde{\pi}_t} \prod_{s=2}^t p_s \prod_{s=1}^t l_s d\bar{\mathbf{x}}_t - \sum_{t'=1}^t \iint \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \left(\frac{\pi_{t'}}{\tilde{\pi}_{t'}} - \frac{\pi_{t'-1}}{\tilde{\pi}_{t'-1}} \right) \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'}.
\end{aligned}$$

For the term within the summation in Equation (S6), we have

$$\begin{aligned}
&\int \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \left(\frac{\pi_{t'}}{\tilde{\pi}_{t'}} - \frac{\pi_{t'-1}}{\tilde{\pi}_{t'-1}} \right) \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} \\
&= \int \{ \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) - m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \} \left(\frac{\pi_{t'}}{\tilde{\pi}_{t'}} - \frac{\pi_{t'-1}}{\tilde{\pi}_{t'-1}} \right) \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'}
\end{aligned}$$

$$\begin{aligned}
& + \int m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \left(\frac{\pi_{t'}}{\tilde{\pi}_{t'}} - \frac{\pi_{t'-1}}{\tilde{\pi}_{t'-1}} \right) \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} \\
& = \int \{ \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) - m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \} \left(\frac{\pi_{t'}}{\tilde{\pi}_{t'}} - \frac{\pi_{t'-1}}{\tilde{\pi}_{t'-1}} \right) \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} \\
& \quad + \int m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \frac{\pi_{t'}}{\tilde{\pi}_{t'}} \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} - \int m_{Y_t S_t}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'-1}) \frac{\pi_{t'-1}}{\tilde{\pi}_{t'-1}} \prod_{s=2}^{t'-1} p_s \prod_{s=1}^{t'-1} l_s d\bar{\mathbf{x}}_{t'-1}, \\
& \quad (S7)
\end{aligned}$$

where the last equality follows from the fact that

$$\int m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) p_{t'} l_{t'} d\bar{\mathbf{x}}_{t'} = m_{Y_t S_t}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'-1}).$$

Substituting Equation (S7) into Equation (S6), we can write Equation (S6) as

$$\begin{aligned}
& \int \frac{m_{Y_t S_t}(\bar{\mathbf{z}}_t, \bar{\mathbf{x}}_t) \pi_t}{\tilde{\pi}_t} \prod_{s=2}^t p_s \prod_{s=1}^t l_s d\bar{\mathbf{x}}_t \\
& - \sum_{t'=1}^t \int \{ \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) - m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \} \left(\frac{\pi_{t'}}{\tilde{\pi}_{t'}} - \frac{\pi_{t'-1}}{\tilde{\pi}_{t'-1}} \right) \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} \\
& - \sum_{t'=1}^t \int \left[m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \frac{\pi_{t'}}{\tilde{\pi}_{t'}} \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} - \int \int m_{Y_t S_t}(\bar{\mathbf{z}}_{t'-1}, \bar{\mathbf{x}}_{t'-1}) \frac{\pi_{t'-1}}{\tilde{\pi}_{t'-1}} \prod_{s=2}^{t'-1} p_s \prod_{s=1}^{t'-1} l_s d\bar{\mathbf{x}}_{t'-1} \right] \\
& = m_{Y_t S_t} + \sum_{t'=1}^t \int \{ m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) - \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \} \left(\frac{\pi_{t'}}{\tilde{\pi}_{t'}} - \frac{\pi_{t'-1}}{\tilde{\pi}_{t'-1}} \right) \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'}.
\end{aligned}$$

Therefore, the bias of the estimator is

$$\sum_{t'=1}^t \int \{ m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) - \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \} \left(\frac{\pi_{t'}}{\tilde{\pi}_{t'}} - \frac{\pi_{t'-1}}{\tilde{\pi}_{t'-1}} \right) \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'},$$

which equals zero when either $(\tilde{m}_{Y_t S_t}, \tilde{m}_{S_t}) = (m_{Y_t S_t}, m_{S_t})$ or $\tilde{\pi} = \pi$. $\square$

S2.5. Proof of Theorem 4. Let $\mathcal{P}_n$ denote the empirical average and $\mathcal{P}$ denote the expectation. For example, $\mathcal{P}_n f(\mathbf{V}) = n^{-1} \sum_{i=1}^n f(\mathbf{V}_i)$ and $\mathcal{P} f(\mathbf{V}) = \mathbb{E}\{f(\mathbf{V}_i)\}$. By the empirical process theory, we have

$$\begin{aligned}
\hat{\phi}_N - \mathbb{E}(\phi_N) &= (\mathcal{P}_n - \mathcal{P})\hat{\phi}_N + \mathcal{P}(\hat{\phi}_N - \phi_N) \\
&= (\mathcal{P}_n - \mathcal{P})\phi_N + \mathcal{P}(\hat{\phi}_N - \phi_N) + o_p(n^{-1/2}),
\end{aligned}$$

where the second equality follows from Condition 2. From the derivation of the bias in the proof of Theorem 3, we have

$$\begin{aligned}
& \mathcal{P}(\hat{\phi}_N - \phi_N) \\
&= \sum_{t'=1}^t \int \{ \hat{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) - \tilde{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) \} \left(\frac{\hat{\pi}_{t'}}{\tilde{\pi}_{t'}} - \frac{\hat{\pi}_{t'-1}}{\tilde{\pi}_{t'-1}} \right) \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{t'=1}^t \int \{\hat{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) - m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'})\} \left(\frac{\hat{\pi}_{t'}}{\pi_{t'}} - \frac{\hat{\pi}_{t'-1}}{\pi_{t'-1}} \right) \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} \\
&\leq \sum_{t'=1}^t \int \{\hat{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) - m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'})\}^2 \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} \int \left(\frac{\hat{\pi}_{t'}}{\pi_{t'}} - \frac{\hat{\pi}_{t'-1}}{\pi_{t'-1}} \right)^2 \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} \\
&\leq C \sum_{t'=1}^t \int \{\hat{m}_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'}) - m_{Y_t S_t}(\bar{\mathbf{z}}_{t'}, \bar{\mathbf{x}}_{t'})\}^2 \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} \int (\hat{w}_{t'} - w_{t'})^2 \prod_{s=2}^{t'} p_s \prod_{s=1}^{t'} l_s d\bar{\mathbf{x}}_{t'} \\
&= o_p(n^{-1/2}).
\end{aligned}$$

where C is a constant, the second equality follows from Condition 1, the first inequality follows from the Cauchy-Schwarz inequality, the second inequality follows from Condition 3, and the last equality follows from Condition 4. As a result, $\hat{\phi}_N(\bar{\mathbf{z}}_t)$ is a semiparametrically efficient for $m_{Y_t S_t}$. Similarly, we can show that $\hat{\phi}_D(\bar{\mathbf{z}}_t)$ is a semiparametrically efficient for m_{S_t} . $\square$

APPENDIX S3: ADDITIONAL RESULTS ON SIMULATION AND EMPIRICAL RESULTS

S3.1. Details of the inference procedure. We use an influence function-based approach with sample splitting to estimate standard errors. Let $\phi(V, \eta)$ denote the influence function of the target quantity, where V represents the observed data and η represents denotes the collection of nuisance functions. We randomly split the full sample $(V_1, V_2, \dots, V_n)$ into K disjoint folds using a random variable S independent of the data, where $S_i \in 1, \dots, K$ indicates the fold assignment for unit i . Let $\hat{\eta}^{(-k)}$ denote the nuisance functions estimated using all data excluding fold k .

The estimated variance of the target parameter is then given by

$$\frac{1}{n} \sum_{i=1}^n \{\phi_i(V_i, \hat{\eta}_{-S_i}) - \phi(V, \hat{\eta})\}^2.$$

Confidence intervals are constructed using a normal approximation based on this variance estimate.

S3.2. Results with time invariant confounders. We maintain the same data generating mechanism as in the main text with continuous outcomes except that the outcomes are now generated from logistic models with the same predictors and coefficients.

Figure S1 shows the absolute biases of the estimators with time invariant confounders and sample size $n = 5,000$ and Figure S2 presents the corresponding RMSEs.

S3.3. Results with time-varying confounders. We consider the same data-generating mechanism as in the main text, except that we fix $\delta = 0.5$ to enforce time-varying confounding. In this setting, Assumption 2 holds with $\mathbf{X}_{i1} = \mathbf{X}_i$, $\mathbf{X}_{i2} = (\mathbf{X}_i, Y_{i1})$, and $\mathbf{X}_{i3} = (\mathbf{X}_i, Y_{i1}, Y_{i2})$.

We first present the results with continuous outcomes. Figure S3 shows the absolute biases of the estimators with time-varying confounders and sample size $n = 5,000$ and Figure S4 presents the corresponding RMSEs.

We then present the results with binary outcomes. Figure S5 shows the absolute biases of the estimators with time invariant confounders and sample size $n = 1,000$ and Figure S6 presents the corresponding RMSEs.

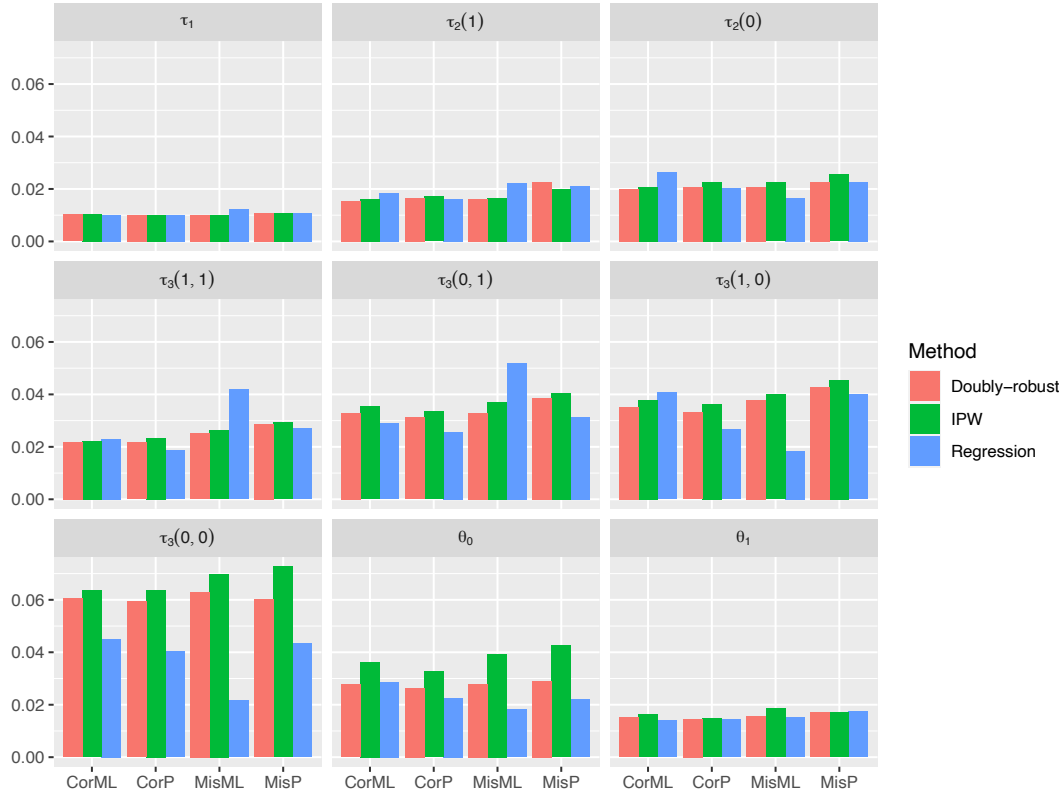


FIG S1. Biases of the estimators across 1000 replications with binary outcomes, $\delta = 0$, and $n = 5,000$. Each row presents the results for one target causal quantity. The column labels “CorML,” “CorP,” “MisML,” and “MisP” stand for the nonparametric estimation with correctly specified covariates, parametric estimation with correctly specified covariates, nonparametric estimation with mis-specified covariates, and parametric estimation with mis-specified covariates, respectively.

TABLE 3

Coverage rates under a naive analysis that ignores selective eligibility, for $\delta = 0$ and $\delta = 0.5$ with $n = 5,000$. The naive analysis treats all cases as independent and applies a standard doubly robust estimator, without accounting for the longitudinal eligibility structure.

	θ_0	θ_1	$\theta_1 - \theta_0$
$\delta = 0$	0.002	0.000	0.000
$\delta = 0.5$	0.000	0.000	0.000

S3.4. Results on coverage rates of the naive analysis.

S3.5. Results on coverage rates with binary outcomes.

S3.6. Analysis of the PSA experiment using naive estimation. To illustrate the consequences of ignoring selective eligibility, we conduct a naive analysis that treats all cases as independent observations and does not account for the longitudinal structure of eligibility. Specifically, we estimate θ_0 , θ_1 , and their difference $\theta_1 - \theta_0$ for each outcome using a standard doubly robust estimator, with nuisance functions estimated nonparametrically. Table 5 presents the results. Across all outcomes, the naive estimates differ substantially from those obtained under our proposed method. These findings reinforce the simulation results in Sec-

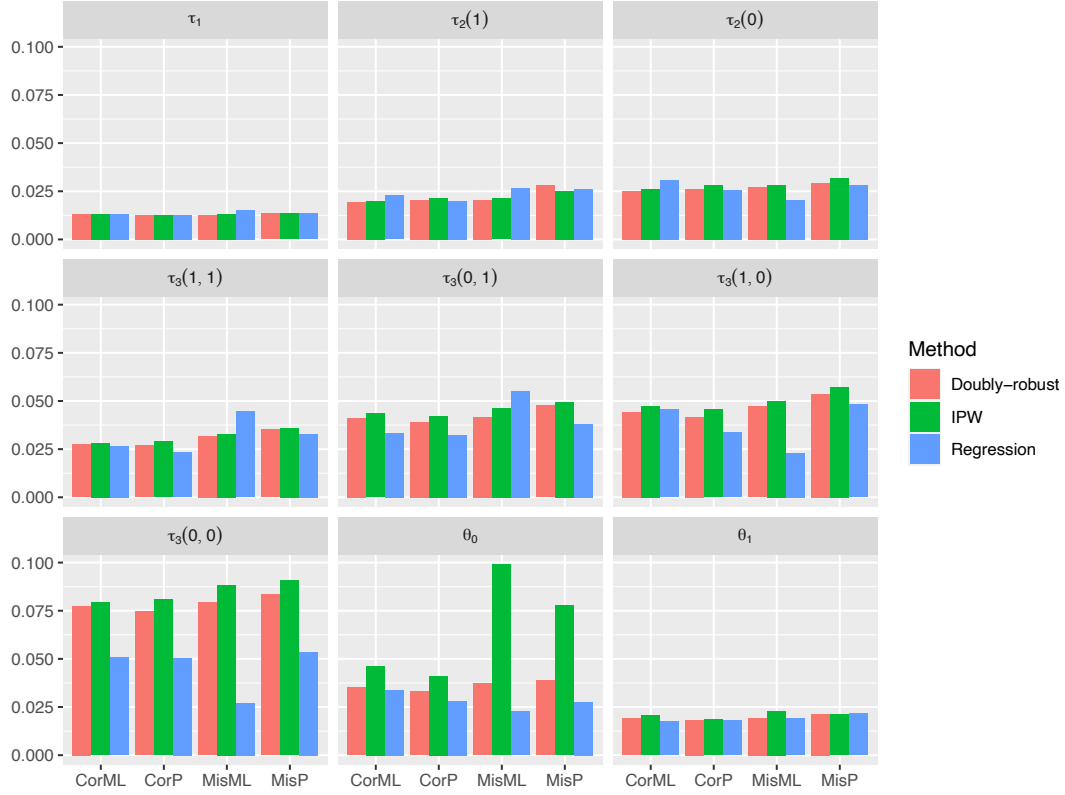


FIG S2. Root mean squared errors for the estimators with binary outcomes, $\delta = 0$, and $n = 5,000$. Each subplot presents the results for one target causal quantity. Within each subplot, the x-axis labels ‘CorML,’ ‘CorP,’ ‘MisML,’ and ‘MisP’ stand for the nonparametric estimation with correctly specified covariates, parametric estimation with correctly specified covariates, nonparametric estimation with mis-specified covariates, and parametric estimation with mis-specified covariates, respectively. The heights of the bars represent the root mean squared errors of the regression (blue), inverse probability weighting (red), and doubly robust estimators (green).

TABLE 4

Coverage rates under different model specifications for $\delta = 0$ and $\delta = 0.5$ with binary outcomes. ‘Cor’ indicates correctly specified covariates, and ‘Mis’ indicates misspecified covariates.

δ		τ_1	$\tau_2(1)$	$\tau_2(0)$	$\tau_3(1, 1)$	$\tau_3(0, 1)$	$\tau_3(1, 0)$	$\tau_3(0, 0)$	θ_0	θ_1
$\delta = 0$	Cor	0.941	0.952	0.957	0.946	0.943	0.948	0.936	0.949	0.939
	Mis	0.941	0.944	0.957	0.896	0.902	0.924	0.937	0.936	0.938
$\delta = 0.5$	Cor	0.961	0.952	0.949	0.953	0.941	0.955	0.934	0.942	0.961
	Mis	0.961	0.948	0.940	0.907	0.908	0.922	0.933	0.950	0.933

tion 4 and underscore the importance of explicitly accounting for selective eligibility in the analyses.

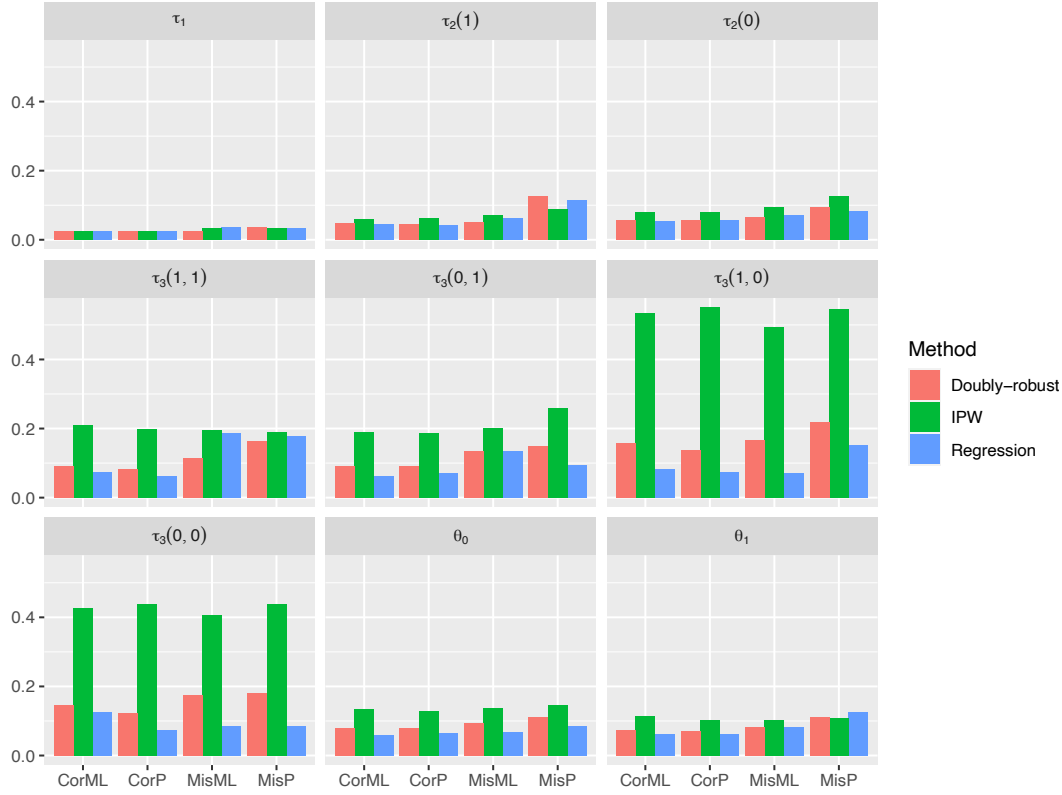


FIG S3. Absolute bias of each estimator across 1,000 replications with $\delta = 0.5$ and $n = 5,000$. Each plot presents the results for one target causal quantity. The labels on the horizontal axis — “CorML,” “CorP,” “MisML,” and “MisP” — stand for the nonparametric estimation with correctly specified covariates, parametric estimation with correctly specified covariates, nonparametric estimation with mis-specified covariates, and parametric estimation with mis-specified covariates, respectively. The height of each bar represents the absolute bias of the regression (blue), IPW (green), or doubly robust estimator (red).

TABLE 5
Naive estimates of θ_0 , θ_1 , and $\theta_1 - \theta_0$ with standard errors in parentheses.

Outcome	θ_0	θ_1	$\theta_1 - \theta_0$
FTA	0.221 (0.008)	0.225 (0.008)	0.004 (0.011)
NCA	0.324 (0.009)	0.345 (0.009)	0.021 (0.012)
NVCA	0.090 (0.005)	0.094 (0.005)	0.004 (0.008)
Agreement	0.668 (0.009)	0.733 (0.008)	0.064 (0.011)
Misclassification (FTA)	-0.390 (0.015)	-0.385 (0.014)	0.004 (0.020)
Misclassification (NCA)	-0.234 (0.016)	-0.200 (0.015)	0.033 (0.022)
Misclassification (NVCA)	-0.604 (0.011)	-0.579 (0.011)	0.026 (0.016)

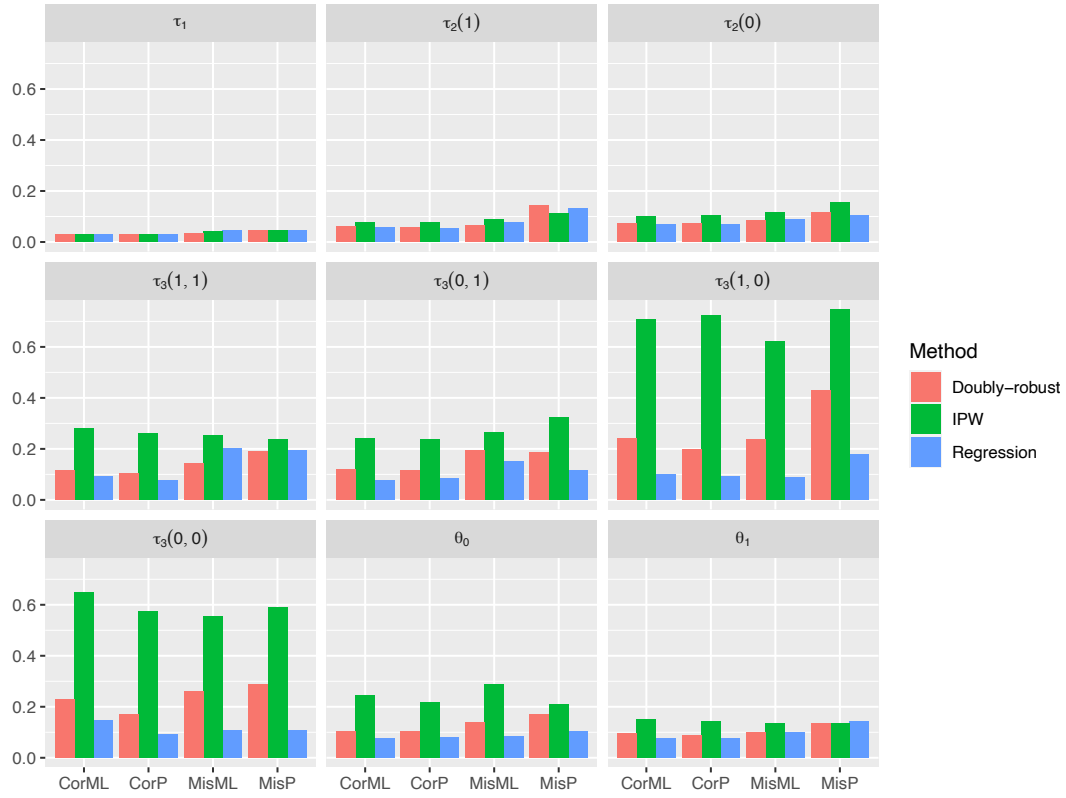


FIG S4. Root mean squared error (RMSE) of each estimator with $\delta = 0.5$ and $n = 5,000$. Each subplot presents the results for one target causal quantity. Within each subplot, the x-axis labels ‘CorML,’ ‘CorP,’ ‘MisML,’ and ‘MisP’ stand for the nonparametric estimation with correctly specified covariates, parametric estimation with correctly specified covariates, nonparametric estimation with mis-specified covariates, and parametric estimation with mis-specified covariates, respectively. The height of each bar represents the RMSE of the regression (blue), IPW (green), or doubly robust estimator (red).

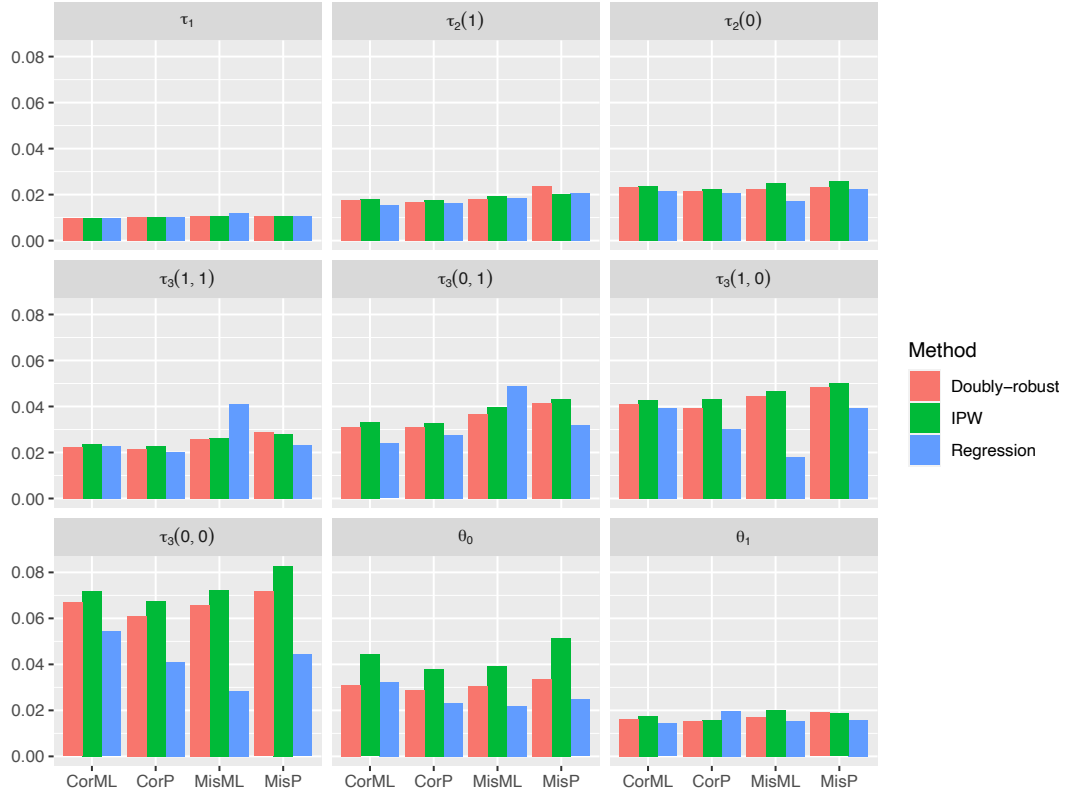


FIG S5. Biases of the estimators across 500 replications with binary outcomes, $\delta = 0.5$, and $n = 5,000$. Each row presents the results for one target causal quantity. The column labels “CorML,” “CorP,” “MisML,” and “MisP” stand for the nonparametric estimation with correctly specified covariates, parametric estimation with correctly specified covariates, nonparametric estimation with mis-specified covariates, and parametric estimation with mis-specified covariates, respectively.

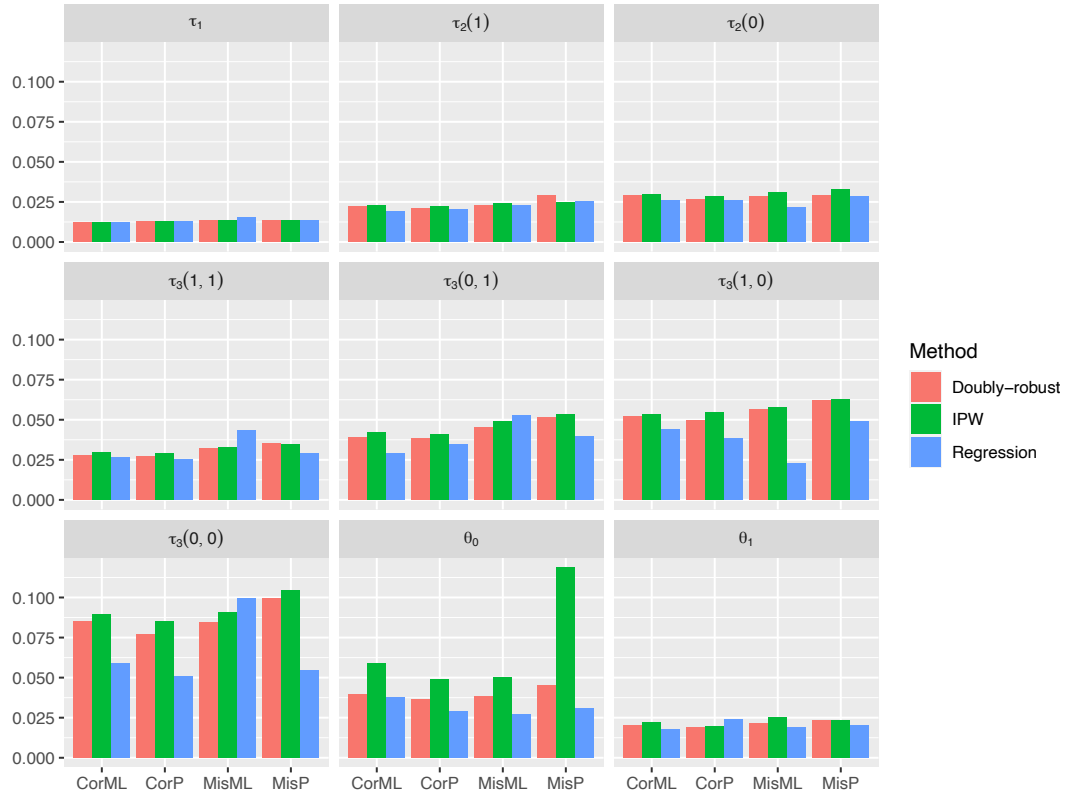


FIG S6. Root mean squared errors for the estimators with binary outcomes, $\delta = 0.5$, and $n = 5,000$. Each subplot presents the results for one target causal quantity. Within each subplot, the x-axis labels ‘CorML,’ ‘CorP,’ ‘MisML,’ and ‘MisP’ stand for the nonparametric estimation with correctly specified covariates, parametric estimation with correctly specified covariates, nonparametric estimation with mis-specified covariates, and parametric estimation with mis-specified covariates, respectively. The heights of the bars represent the root mean squared errors of the regression (blue), inverse probability weighting (red), and doubly robust estimators (green).